OPTIMUM DESIGN OF STIFFENED CYLINDRICAL AND CONICAL SHELLS

A Thesis Submitted
in Partial Fulfilment of the Requirements
for the Degree of
MASTER OF TECHNOLOGY

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to the
DEPARTMENT OF MECHANICAL ENGINEERING
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AUGUST, 1978

ME-1978-M-RED-OPT

CERTIFICATE

This is to certify that the thesis entitled "Optimum Design of Stiffened Cylindrical and Conical Shells" is a record of work carried out under my supervision and that it has not been submitted elsewhere for a degree.

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ACKNOWLEDGEMENT

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NOMENCLATURE

A   —   Cross-sectional area of stiffner

a   —   Axial length of cylindrical shell

D   —   Extensional rigidity

d   —   Spacing of stringers

E   —   Young's modulus

F   —   Resonant frequency (Hz)

f   —   Objective function (weight of shell)

G   —   Shear modulus

g   —   Acceleration due to gravity

I   —   Moment of inertia of stiffener about its centroid

I_o —   Moment of inertia of stiffener about middle surface of cylinder wall.

J   —   Torsional constant

K   —   Bending rigidity

l   —   Spacing of rings

M   —   Mass per unit area of cylindrical shell

m   —   Number of axial/meridional half waves

N_x, N_y, N_{xy} —   Stress resultants in cylindrical shell

N_θ, N_s, $N_{\theta s}$ —   Stress resultants in conical shell

n   —   Number of circumferential full waves

n_r —   Number of rings

n_s —   Number of stringers

P   —   Axial load

R — Mean radius of cylindrical shell

R_1 — Mean radius of cone at the small end

R_2 — Mean radius of cone at the large end

$\bar{R}$ — Non-dimensional parameter, $\dfrac{E_r A_r}{E t_w l}$

S — Rigidity moment

$\bar{S}$ — Non dimensional parameter, $\dfrac{E_s A_s}{E t_w d}$

s — Distance from cone apex along meridion to a point on shell middle surface

s_1, s_2 — Distance along meridion from cone apex to small base and large base of conical shell respectively

T — Twisting rigidity

t — Time (secs.)

$\bar{t}$ — Effective wall thickness of shell, $\dfrac{A_s}{d} + t_w$

t_1 — Thickness of ring

t_2 — Thickness of stringer

t_w — Thickness of shell wall

u, v, w — Displacements of a point on shell middle surface in x, y, z or θ, s, z directions, respectively

$\bar{u}$, $\bar{v}$, $\bar{w}$ — Displacement amplitudes

x, y, z — Orthogonal coordinates as defined in Fig. 2.1 (x and y lie in the middle surface of cylinder)

Z — Curvature parameter, $\dfrac{a^2}{R t_w} (1 - \mu^2)^{1/2}$

$\bar{z}$ — Distance of centroid of stiffener to middle surface of shell, (positive if stiffener lies on external surface of shell)

θ, s, z — Orthogonal coordinates as defined in Fig. 3.1

(θ and s lie in the middle surface of cone)

Greek Symbols

α, β — Wave length parameters or angles (Fig. 3.1)

γ — Specific weight

$\epsilon_x, \epsilon_y, \epsilon_{xy}$ — Middle surface strains
$\epsilon_\theta, \epsilon_s, \epsilon_{\theta s}$

μ — Poisson's ratio

Π — Potential energy or change in

potential energy

χ — Curvature

ρ — Mass density

ω — Circular frequency (rad/sec)

Subscripts

c — Cylinder or cone

r — Ring

s — Stringer

ω — Inertia load

w — Shell wall

A subscript preceded by a comma indicates

partial differentiation with respect to the

subscript.

SYNOPSIS

In the present work, the design optimization of axially loaded, simply supported stiffened cylindrical and conical shells for minimum weight is carried. The design variables are thickness of shell wall, thicknesses and depths of rings and stringers, number/spacing of rings and number/spacing of stringers. Natural frequency, overall buckling strength and direct stress constraints are considered in both cylindrical and conical shell design problems. In the case of cylindrical shell the local buckling constraints are also included.

The design of cylindrical shell is carried with three different combinations of stiffeners. In the case of conical shell, an expression for the buckling load is derived and optimum designs are obtained with rectangular stiffeners. For all the examples, the independent effects of behaviour constraints are also studied separately. The optimum designs are achieved with one of the standard nonlinear constrained optimization techniques (Davidon - Fletcher - Powell method with interior penalty function formulation) and few optimal solutions are checked for the satisfaction of Kuhn-Tucker conditions.

CHAPTER I

INTRODUCTION

1.1 IMPORTANCE OF THE PROBLEM:

One of the most common structural elements used in
modern airplane, missile, booster, and other aerospace
vehicles is the thin conical or cylindrical shell. In
aerospace vehicles these shells are used as interstage
adapters, which mate the space-craft to the rocket vehicle.
These shells are weak in resisting the frequently encoun-
tered high compressive stresses. Hence to achieve higher
efficiencies, it is necessary to design the shells with
high buckling strength-to-weight ratio while satisfying the
minimum natural frequency and yield strength requirements.
One of the common means of achieving the above requirements
is to stiffen the shell by longitudinal (stringer) and
circumferential (ring) stiffeners. Thus the design problem
becomes one of finding the skin thickness of the shell,
sizes of the stiffeners, and the number of rings and strin-
gers.

1.2 STATE OF ART

Overall buckling and free vibration analyses of
stiffened cylindrical and conical shells have been of
interest to designers for a long time. Because of the

complexity in the analyses, earlier designs are based on experimental data and local buckling equations. But the data has wide scatter and hence one has to make a conservative estimate of the critical values of buckling load and natural frequencies. The designs obtained from these critical values may not be optimum. However, some work has been done by using approximate analyses.

The buckling analysis of cylindrical shell has been made by many investigators [1 - 12]. Serpico [1] and Weingarten et al. [2, 3, 4] have studied cylindrical and conical shells without stiffeners. Later the effect of stiffeners has been considered in the analysis by Block et al. [5] and Singer et al. [6]. In Ref. [5] and [6], the effect of stiffeners has been averaged over the shell surface. A few research workers [8, 9, 12] have accounted for the discreteness of stiffeners.

Free vibration analysis of stiffened cylindrical [13 - 15] and conical [16, 17] shells has been carried by few research workers. In Ref. [13] an expression for the natural frequencies has been derived, considering the effect of stiffeners by smearing technique. Later Egle and Sewall [14] have analysed cylindrical shells accounting for the discreteness of the stiffeners. Crenwelge and Muster [17], using the energy approach, indicated a method to determine the natural frequencies of simply

supported conical shells in unstiffened, ring-stiffened
or ring-and-stringer stiffened configurations.

A wealth of literature [18 - 38] has been devoted
for the optimum design of cylindrical shells subjected to
general and local buckling constraints. Before 1967, the
studies [21 - 24] were based on the assumption of simulta-
neous failure modes and the optimization was achieved by
parametric studies. Afterwards optimum design has been
achieved by the use of constrained optimization techniques.
Pappas and Amba Rao[30] have considered the designs with
conventional as well as spiral stiffeners. In 1975,
Fronowicki et al. [38] presented the design of shell with T -
ring stiffeners taking three different objectives subjected
to vibration constraint. The three objectives are (i) mini-
mum weight of the stiffened shell, (ii) maximum separation
of the lowest two natural frequencies, and (iii) maximum
separation of the lowest two natural frequencies which have
primarily axial contents.

The design of stiffened conical shells has been
attempted by only few research workers. In 1966, Burns
[39] has achieved optimum design of ring-stiffened shells
by parametric study. The loading considered was hydro-
static compression. Later Thornton[40] and Herald et al.
[41] have used nonlinear programming techniques for the
design of stiffened conical shells with buckling and
yield stress constraints.

1.3 OBJECT AND SCOPE OF PRESENT WORK

The object of this work is to present a unified approach for the design optimization of simply supported stiffened cylindrical and conical shells for minimum weight. Axial loading is assumed for both the shells. For conical shells, an expression for critical buckling load is derived. In the case of cylindrical shell, examples with three different combinations of stiffeners are considered. An example which was solved in Ref. [26] is also solved with the constraint set derived in this work and the results are compared. Only one type of stiffeners (rectangular) are considered in the case of conical shell. In all the examples, the effects of frequency and buckling constraints are studied separately and the results are compared. The examples are formulated as standard constrained optimization problems and are solved by gradient method (Davidon - Fletcher - Powell method) with interior penalty function formulation. Finite difference (backward differences) technique is used for the gradient calculations. The optimum solutions have been verified for the satisfaction of Kuhn - Tucker conditions in few cases.

1.4 ORGANIZATION OF THE THESIS

The thesis is divided into six chapters. In second and third chapters, the buckling and vibration analyses of stiffened cylindrical and conical shells are presented respectively. The formulation of the optimization problem and solution procedure are given in chapter four. Results are presented and discussed in chapter five. Chapter six gives the conclusions of the present study and recommendations for future work.

CHAPTER 2

ANALYSIS OF CYLINDRICAL SHELL

The buckling and vibration analysis of stiffened cylindrical shell, presented in this chapter, has been originally given by Block et al.[5] and Mikulas et al.[13] based on energy principles. Their analysis is based on small deflection theory and includes eccentricity (one - sided) effects of the stiffeners. The results of this analysis have been shown to be in good agreement with other theoriesand experimental results. The stiffened cylinder is considered to be composed of an isotropic shell stiffened by uniform, equally spaced rings and stringers, all having elastic material properties.

The following assumptions have been made in this analysis:

(i) the rings and the stringers are spaced closely so that their elastic and inertial properties can be averaged over the stiffener spacing,

(ii) the usual Donnell type assumptions are used to specify the displacements in the shell, where as the stiffeners are treated as beam elements with stiffener twisting accounted for in an approximate manner,

(iii) in cases where rings and stringers lie on the same surface of the shell, the effect of the joints of the stiffener frame is ignored.

2.1 BUCKLING ANALYSIS

For the coordinate system shown in Fig. 2.1, the strain-displacement relations in the shell due to buckling displacements can be written as

$$\varepsilon_x = \bar{\varepsilon}_x - z\, w,_{xx}$$
$$\varepsilon_y = \bar{\varepsilon}_y - z\, w,_{yy} \tag{2.1}$$
$$\varepsilon_{xy} = \bar{\varepsilon}_{xy} - 2z\, w,_{xy}$$

with

$$\bar{\varepsilon}_x = u,_x$$
$$\bar{\varepsilon}_y = v,_y + \frac{w}{R}$$
$$\bar{e}_{xy} = u,_y + v,_x$$

where u, v, and w are the displacements induced by buckling.

If the stiffeners are assumed to behave as beam elements, the stiffener strain,- displacement relations can be written as

$$\varepsilon_{x_s} = \bar{\varepsilon}_x - z\, w,_{xx}$$
$$\varepsilon_{y_r} = \bar{\varepsilon}_y - z\, w,_{yy} \tag{2.2}$$

where the subscripts s and r are used to denote stringers and rings, respectively. Equation (2.2) specifies that buckling displacements are such that the strain varies linearly across the depth of the stiffner, yet satisfies the compatibility of displacements between the stiffener and the surface of the shell to which it is fastened. This compatibility requirement is the source of eccentricity or one-sided effect in the buckling of stiffened shells.

The strain energy of the isotropic shell π_c can be expressed in terms of middle surface displacements as

$$\pi_c = \frac{E\, t_w}{2\,(1 - \mu^2)} \int_0^{2\pi R} \int_0^a [\, u_{,x}^2 + (v_{,y} + \frac{W}{R})^2$$

$$+ 2\,\mu\, u_{,x}\,(v_{,y} + \frac{W}{R}) + \frac{(1 - \mu)}{2}\,(u_{,y} + v_{,x})^2\,]\ dx\ dy$$

$$+ \frac{D}{2} \int_0^{2\pi R} \int_0^a [\, w_{,xx}^2 + w_{,yy}^2 + 2\, w_{,xx}\, w_{,yy}$$

$$+ 2\,(1 - \mu)\, w_{,xy}^2\,]\ dx\ dy \qquad\qquad (2.3)$$

where

$$D = \frac{E\, t_w^3}{12\,(1 - \mu^2)}$$

The strain energies of stringers π_s and of rings π_r corresponding to the buckling displacements can be expressed as

$$\pi_s = \frac{1}{2d} \int_0^{2\pi R} \int_0^a [E_s A_s u_{,x}^2 - 2 \bar{z}_s E_s A_s u_{,x} w_{,xx}$$

$$+ E_s Io_s w_{,xx}^2 + G_s J_s w_{,xy}^2] \, dx \, dy \qquad (2.4)$$

and

$$\pi_r = \frac{1}{2l} \int_0^{2\pi R} \int_0^a [E_r A_r (v_{,y} + \frac{w}{R})^2 - 2 \bar{z}_r E_r A_r$$

$$(v_{,y} + \frac{w}{R}) w_{,yy} + E_r Io_r w_{,yy}^2 + G_r J_r w_{,xy}^2] \, dx \, dy$$

$$(2.5)$$

where d is the stringer spacing and l is the ring spacing. The subscripts s and r are used to denote stringer and ring properties, respectively. $\bar{z}$ denotes the distance from the centroid of a stiffener to the middle surface of the shell as shown in Fig. 2.1, and is positive if the stiffeners are on the external surface of the cylinder. Io is the moment of inertia of a stiffener about the middle surface of the shell.

In Eqs. (2.4) and (2.5), the energy contributions of the stringers and the rings are averaged over the circumference and the length of the cylinder respectively; and hence, discreteness of the stiffeners is ignored.

A rigorous derivation of the buckling equations from the potential energy of the loaded shell requires the

use of nonlinear theory and techniques. However, when the effects of prebuckling deformations are not considered explicitly, the potential energy of the external forces causing buckling can be derived as follows.

As the middle surface of the shell bends due to buckling, no membrane stresses are produced but the edges get closer to each other which give rise to inplane displacements. The inplane displacement of an element of length dx in x - direction can be obtained from Fig. 2.2, as

$$\Delta u = ds - dx = \frac{1}{2} \left(\frac{\partial w}{\partial x} \right)^2 dx \tag{2.6a}$$

Consequently, the work done due to the force N_x acting on an element of width dy of shell element can be expressed as

$$N_x \Delta u = \frac{1}{2} N_x \left(\frac{\partial w}{\partial x} \right)^2 dx \tag{2.6b}$$

Similarly, considering the work done by the forces N_y and N_{xy}, the potential energy of the external forces is

$$\pi_1 = -\frac{1}{2} \int_o^{2\pi R} \int_o^a [N_x w_{,x}^2 + 2 N_{xy} w_{,x} w_{,y}$$

$$+ N_y w_{,y}^2] dx \, dy \tag{2.6}$$

where N_x is a stress resultant (positive in compression) obtained by considering the shell and stringers to be loaded with a uniform normal stress in the x - direction.
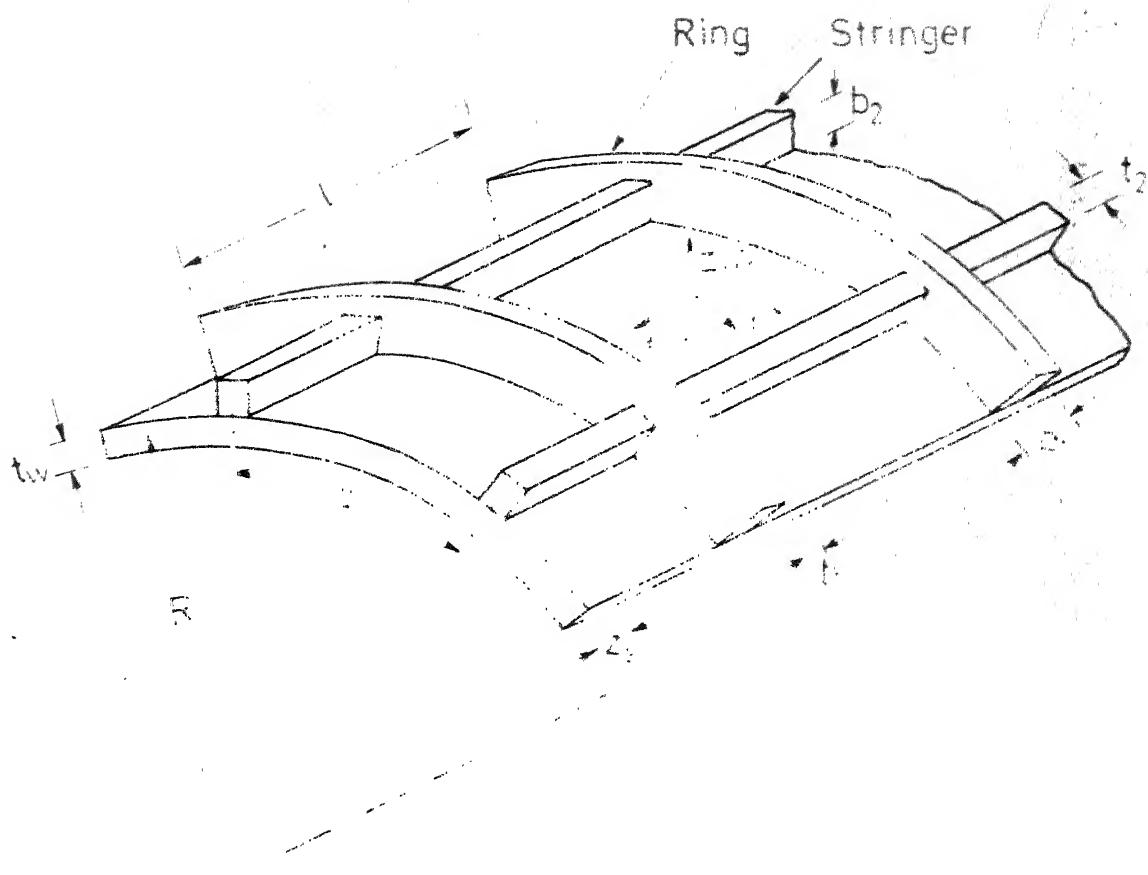
FIG. 2·1 GEOMETRY AND COORDINATE SYSTEM
OF CYLINDRICAL SHELL WITH STIFFENERS

$$\epsilon_x = \frac{ds-dx}{dx} = \sqrt{1+\left(\frac{\partial w}{\partial x}\right)^2} - 1 \approx \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$
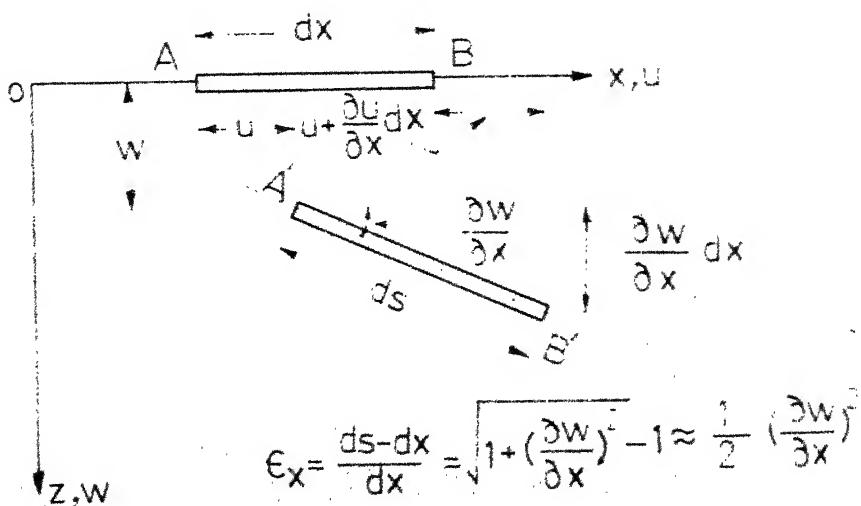
FIG. 2·2 STRAIN, ϵ_x, DUE TO LARGE
DEFLECTION (w)

Similarly N_y is a stress resultant (positive in compression) obtained by considering the shell and rings to be loaded with a normal stress in the y - direction.

The total potential energy of the loaded stiffened cylinder, π_T can be written as

$$\pi_T = \pi_c + \pi_s + \pi_r + \pi_l \tag{2.7}$$

The application of the principle of minimum potential energy ($\delta \pi_T = 0$) yields the equilibrium equations and the boundary conditions corresponding to arbitrary variations in u, v and w respectively. The displacement functions which satisfy the simply supported boundary conditions $\left[v\ (x=0,y) = v\ (x=a,y) = 0 \text{ and } w\ (x=0,y) = w\ (x=a,y) = 0 \right]$ are

$$u = \bar{u}\ \cos\ \frac{m \pi x}{a}\ \cos\ \frac{ny}{R}$$

$$v = \bar{v}\ \sin\ \frac{m \pi x}{a}\ \sin\ \frac{ny}{R} \tag{2.8}$$

$$w = \bar{w}\ \sin\ \frac{m \pi x}{a}\ \cos\ \frac{ny}{R}$$

where m denotes the number of longitudinal half waves and n indicates the number of circumferential full waves in the buckled pattern.

If Eq. (2.8) is substituted into the equilibrium equations obtained from $\delta \pi_T = 0$ with $N_{xy} = 0$, the existence of nontrivial buckling displacements require that the determinant of coefficient matrix of $\bar{u}$, $\bar{v}$ and $\bar{w}$

vanish. This condition reduces to the stability equation:

$$(N_x + N_y \; \beta^2) \; \frac{a^2}{\pi^2 \, D} \; = \; m^2 \, (1 + \beta^2) + m^2 \; \frac{E_s \; I_s}{d \; D} + m^2 \beta^4 \; \frac{E_r \; I_r}{l \; D}$$

$$+ \; (\; \frac{G_s \; J_s}{d \; D} + \frac{G_r \; J_r}{l \; D} \;) \; m^2 \beta^2$$

$$+ \; \frac{12 \, Z^2}{m^2 \pi^4} \; [\; \frac{1 + \bar{S} \, \Lambda_s + \bar{R} \, \Lambda_r + \bar{S} \, \bar{R} \; \Lambda_{rs}}{\Lambda} \;] \qquad (2.9)$$

where

$$\Lambda_r \; = \; 1 + 2 \, \alpha^2 \, \beta^2 \, (1 - \beta^2 \mu) \, (\; \frac{\bar{z}_r}{R} \;) + \alpha^4 \beta^4 \, (1 + \beta^2)^2 \, (\frac{\bar{z}_r}{R})^2$$

$$\Lambda_s \; = \; 1 + 2 \alpha^2 \, (\; \beta^2 - \mu) \, (\; \frac{\bar{z}_s}{R}) + \alpha^4 \, (1 + \beta^2)^2 \, (\; \frac{\bar{z}_s}{R})^2$$

$$\Lambda_{rs} \; = \; 1 - \mu^2 + 2 \, \alpha^2 \, \beta^2 \, (1 - \mu^2) \, (\; \frac{\bar{z}_r}{R} + \frac{\bar{z}_s}{R} \;)$$

$$+ \alpha^4 \beta^4 \, [1 - \mu^2 + 2 \, \beta^2 \, (1 + \mu) \,] \; (\; \frac{\bar{z}_r}{R})^2$$

$$+ \; 2 \, \alpha^4 \, \beta^4 \, (1 + \mu)^2 \, (\; \frac{\bar{z}_r}{R} \;) \, (\; \frac{\bar{z}_s}{R} \;) + \alpha^4 \, \beta^2$$

$$[\; 2 \, (1 + \mu) + \beta^2 \, (1 - \mu^2) \,] \; (\; \frac{\bar{z}_s}{R} \;)^2$$

and

$$\Lambda = \; (1 + \beta^2)^2 + 2 \, \beta^2 \, (1 + \mu) \, (\; \bar{R} + \bar{S} \;)$$

$$+ (1 - \mu^2) \, [\; \bar{S} + 2 \, \beta^2 \, \bar{R} \, \bar{S} \, (1 + \mu) + \beta^4 \, \bar{R} \;]$$

with

$$Z^2 \; = \; \frac{a^4 \, (1 - \mu^2)}{R^2 \, t_w^2} \qquad\qquad D \; = \; \frac{E \, t_w^3}{12 \, (1 - \mu^2)}$$

$$\overline{S} = \frac{E_s \, A_s}{E \, t_w \, d} \qquad\qquad \overline{R} = \frac{E_r \, A_r}{E \, t_w \, l}$$

$$\gamma = \frac{m \, \pi \, R}{a} \qquad\qquad \beta = \frac{n \, a}{m \, \pi \, R}$$

In Eq. (2.9), the specified combination of axial and circumferential loading must be minimized numerically for integral value of m and n in order to compute the critical buckling load.

2.2 FREE VIBRATION ANALYSIS

In the case of free vibration analysis, the total potential energy π_T of the stiffened cylindrical shell is composed of strain energy of the shell, π_c, strain energy of the stringers, π_s, strain energy of the rings, π_r, and potential energy of the inertia loading, π_ω. The expressions for π_c, π_s and π_r are the same as defined in Section 2.1 [Eqs. (2.3), (2.4) and (2.5)]. An expression for π_ω can be obtained as follows.

If the shell is undergoing simple harmonic motion with circular frequency ω (inplane inertia neglected), and w (x, y) is the deflection shape at the time of maximum deflection, the potential energy due to inertia load can be written as

$$\pi_\omega = -\frac{1}{2} \int_0^{2\pi R} \int_0^a M \, \omega^2 \, w^2 \, dx \, dy \qquad (2.10)$$

where $M = \rho_w t_w + \rho_s \dfrac{A_s}{d} + \rho_r \dfrac{A_r}{l}$, is the average smeared-out mass per unit area of the shell. The quantities ρ_w, ρ_s and ρ_r are the mass densities of cylinder wall, stringers and rings, respectively.

The application of Hamilton's principle or principle of minimum potential energy in equivalent static case ($\delta \pi_T = 0$) yields three equilibrium equations. On substituting the displacement solution, given by Eq. (2.8), in the equilibrium equations, a set of homogeneous linear equations is obtained in $\bar{u}$, $\bar{v}$ and $\bar{w}$. Applying the condition for nontrivial solution, an expression for frequency is obtained as

$$\frac{4 M a^4 F^2}{\pi^2 D} = m^4 (1 + \beta^2)^2 + m^4 [\frac{E_s I_s}{d D}$$

$$+ \beta^2 (\frac{G_s J_s}{d D} + \frac{G_r J_r}{l D}) + \beta^4 \frac{E_r I_r}{l D}]$$

$$+ \frac{12 Z^2}{\pi^4} [\frac{1 + \bar{S} \Lambda_s + \bar{R} \Lambda_r + \bar{S} \bar{R} \Lambda_{rs}}{\Lambda} \} \qquad (2.11)$$

where

F is the frequency in cycles/sec.

$$\Lambda_s = 1 + 2\alpha^2 (\frac{\bar{z}_s}{R}) (\beta^2 - \mu) + \alpha^4 (\frac{\bar{z}_s}{R})^2 (1 + \beta^2)^2$$

$$\Lambda_r = 1 + 2 n^2 (\frac{\bar{z}_r}{R}) (1 - \beta^2 \mu) + n^4 (\frac{\bar{z}_r}{R})^2 (1 + \beta^2)^2$$

$$\Lambda_{rs} = 1 - \mu^2 + n^2 \alpha^2 [\beta^2 (1 - \mu^2) + 2 (1 + \mu)] \left(\frac{\bar{z}_s}{R} \right)^2$$

$$+ n^4 [1 - \mu^2 + 2 \beta^2 (1 + \mu)] \left(\frac{\bar{z}_r}{R} \right)^2$$

$$+ 2 n^2 (1 - \mu^2) [\left(\frac{\bar{z}_s}{R} \right) + \left(\frac{\bar{z}_r}{R} \right)]$$

$$+ 2 n^4 (1 + \mu)^2 \left(\frac{\bar{z}_r}{R} \right) \left(\frac{\bar{z}_s}{R} \right)$$

and

Λ, z^2, $\bar{S}$, $\bar{R}$, α, and β are same as defined in Eq. (2.9).

In order to get the minimum frequency, the right hand side of Eq. (2.11) must be minimized numerically for integral values of m and n.

2.3 STRESS ANALYSIS

(i) Stress in Shell Wall:

For an axially loaded stiffened cylindrical shell, the stress induced in the shell wall can be taken approximately as

$$\sigma_w = \frac{P}{2 \pi R \bar{t}} \tag{2.12}$$

where P is the axial load,

R is the mean radius of the shell,

and $\bar{t}$ is the effective thickness of the stiffened shell in axial direction [43], ($\bar{t} = \frac{A_s}{d} + t_w$).

(ii) Local Buckling Stresses:

If the stringer spacing is small relative to the radius of the cylinder, classical flat plate theory can be used to predict local instability in the cylinder wall (panel buckling)[22].

The critical buckling stress in the cylinder wall is given by

$$\sigma cr_w = [4\pi^2 E_w/12(1-\mu^2)] \left(\frac{t_w}{d}\right)^2 \tag{2.13}$$

Similarly, local instability in rectangular stringers can be predicted using the following equation [22]:

$$\sigma cr_s = [0.5\pi^2 E_s/12(1-\mu^2)] \left(\frac{t_2}{l}\right) \tag{2.14}$$

where

t_2 is thickness of stringer and

l is spacing of rings.

If stringers are considered to fail in the column buckling mode, the critical load is given by

$$P_{cr_s} = 0.5 \frac{\pi^2 E_s I_s}{l^2} \tag{2.15}$$

where l is length of the stringer between two rings
(equal to ring spacing),

and I_s is moment of inertia of stringer about its
centroidal axis.

CHAPTER 3

ANALYSIS OF CONICAL SHELL

In this chapter, the buckling and free vibration analysis of stiffened conical shell is presented. The shell is assumed to be loaded with an intensity of P (load per unit length of circumference) at the small end in the axial direction. For this case, an expression for the buckling load intensity is derived in Section 3.1. In Section 3.2, the free vibration analysis is presented as per the method indicated by Crenwelge and Muster [17]. The results of free vibration analysis have been shown to be in good agreement with other theories and experimental results [17].

The following assumptions are made in the buckling and vibration analysis of conical shell:

   (i)   shell displacements and displacement gradients are small so that linear shell theory is applicable,

   (ii)   rings and stringers are closely spaced so that their elastic and inertial properties can be averaged over their respective spacings,

   (iii)   the stiffeners are thin, the interactions at the ring and stringer joints may be ignored,

   (iv)   stiffeners are twisted about their centroids.

3.1 BUCKLING ANALYSIS

Consider a thin truncated conical shell, whose geometry and coordinates are shown in Fig. 3.1. The strains at a distance z from the middle surface of the shell are given by

$$\epsilon_s = \bar{\epsilon}_s - z \chi_s$$
$$\epsilon_\theta = \bar{\epsilon}_\theta - z \chi_\theta \tag{3.1}$$
$$\epsilon_{s\theta} = \bar{\epsilon}_{s\theta} - 2z \chi_{s\theta}$$

For the rings

$$\epsilon_{\theta_r} = \bar{\epsilon}_\theta - z \chi_\theta$$
$$\chi_{s\theta_r} = w,_{s\theta} / (s \cos\alpha) \tag{3.2}$$

and for the stringers

$$\epsilon_{s_s} = \bar{\epsilon}_s - z \chi_s \tag{3.3}$$
$$\chi_{s\theta_s} = \frac{w,_{s\theta}}{s \cos\alpha} - \frac{2 w,_\theta}{s^2 \cos\alpha} - \frac{u,_s \tan\alpha}{s} + \frac{2 u \tan\alpha}{s^2}$$

where

$$\bar{\epsilon}_s = v,_s$$

$$\bar{\epsilon}_\theta = (u,_\theta / \cos\alpha + v + w \tan\alpha) / s$$

$$\bar{\epsilon}_{s\theta} = u,_s - u/s + v,_\theta / (s \cos\alpha)$$

$$\chi_s = w,_{ss}$$

$$\chi_\theta = \frac{w,_s}{s} + (w,_{\theta\theta} - u,_\theta \sin\alpha) / (s^2 \cos^2\alpha)$$

$$\chi_{s\theta} = \frac{w,_{s\theta}}{s\cos\alpha} - \frac{w,_{\theta}}{s^2\cos\alpha} - \frac{u,_s\tan\alpha}{2\,s} + \frac{u\tan\alpha}{s^2} ,$$

and u, v and w indicate buckling displacements.

The strain energies of cone, stringers and rings in terms of strains are given by

$$\chi\ldots\Pi_c = \frac{1}{2} \int_{s_1}^{s_2} \int_{0}^{2\pi} [\, D_w \{\, \bar{\varepsilon}_\theta^2 + \bar{\varepsilon}_s^2 + 2\mu_w \bar{\varepsilon}_s \bar{\varepsilon}_\theta + \frac{1}{2}(1-\mu_w)$$

$$\bar{\varepsilon}^2_{s\theta}\} + K_w\{\chi_\theta^2 + \chi_s^2 + 2\mu_w\chi_s\chi_\theta + 2(1-\mu_w)\chi_{s\theta}^2\}\,]$$

$$s\cos\alpha\; d\theta\, ds \qquad\qquad (3.4)$$

$$\pi_s = \frac{1}{2} \int_{s_1}^{s_2} \int_{0}^{2\pi} [\, D_s \bar{\varepsilon}_s^2 - 2 S_s \bar{\varepsilon}_s \chi_s + K_s \chi_s^2$$

$$+ T_s \chi^2_{s\theta_s}\,]\; s\cos\alpha\; d\theta\, ds \qquad (3.5)$$

$$\pi_r = \frac{1}{2} \int_{s_1}^{s_2} \int_{0}^{2\pi} [\, D_r \bar{\varepsilon}_\theta^2 - 2 S_r \bar{\varepsilon}_\theta \chi_\theta + K_r \chi_\theta^2 + T_r \chi_{s\theta_r}^2\,]$$

$$s\cos\alpha\; d\theta\, ds \qquad\qquad (3.6)$$

where

$$D_w = E_w t_w / (1-\mu_w^2)$$

$$D_r = E_r A_r / b_r$$

$$D_s = E_s A_s / b_s$$

$$S_r = E_r A_r \bar{z}_r / b_r$$

$$S_s = E_s A_s \bar{z}_s / b_s$$

$$K_w = E_w t_w^3 / [12 (1 - \mu_w^2)]$$

$$K_r = E_r (I_r + A_r \bar{z}_r^2) / b_r$$

$$K_s = E_s (I_s + A_s \bar{z}_s^2) / b_s$$

$$T_r = G_r J_r / b_r$$

$$T_s = G_s J_s / b_s$$

and $b_s = s \beta \cos \alpha$

The potential energy of the external forces causing buckling is given by

$$\pi_1 = -\frac{1}{2} \int_{s_1}^{s_2} \int_0^{2\pi} [N_\theta w,_\theta^2 + 2 N_{\theta s} w,_\theta w,_s + N_s w,_s^2]$$

$$s \cos \alpha \ d\theta \ ds \qquad (3.7)$$

For an axial load (compression) of intensity P (load per unit length around the circumference), the stress resultants N_θ , $N_{s\theta}$ and N_s are given by[42]

$$N_\theta = 0$$
$$N_{s\theta} = 0 \qquad (3.8)$$
$$N_s = P \frac{s_1}{s \sin \alpha}$$

The total potential energy of the shell can be expressed as

$$\pi_T = \pi_c + \pi_s + \pi_r + \pi_1 \qquad (3.9)$$
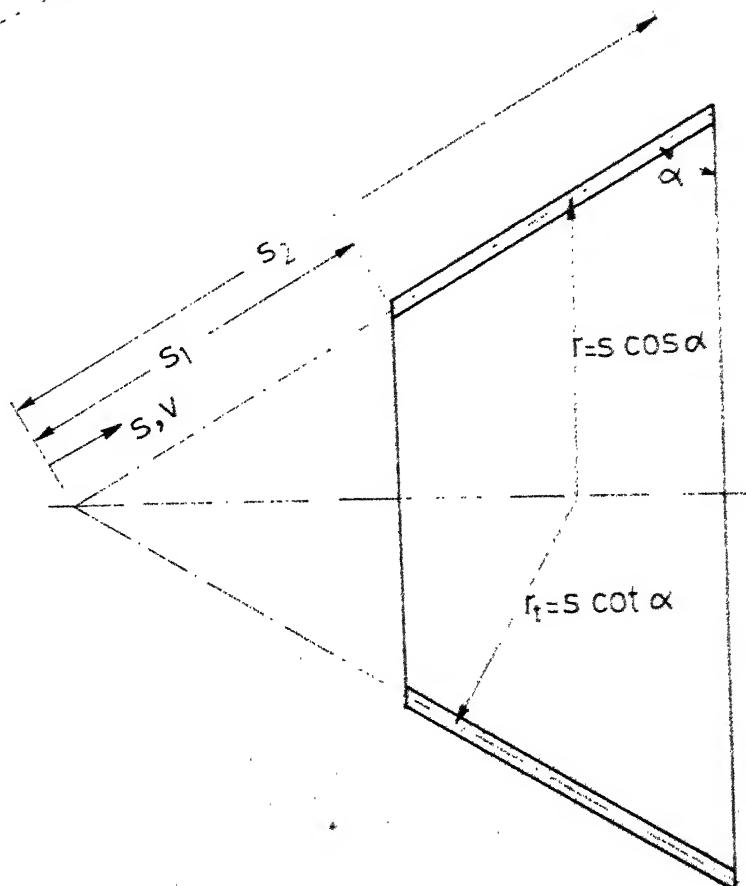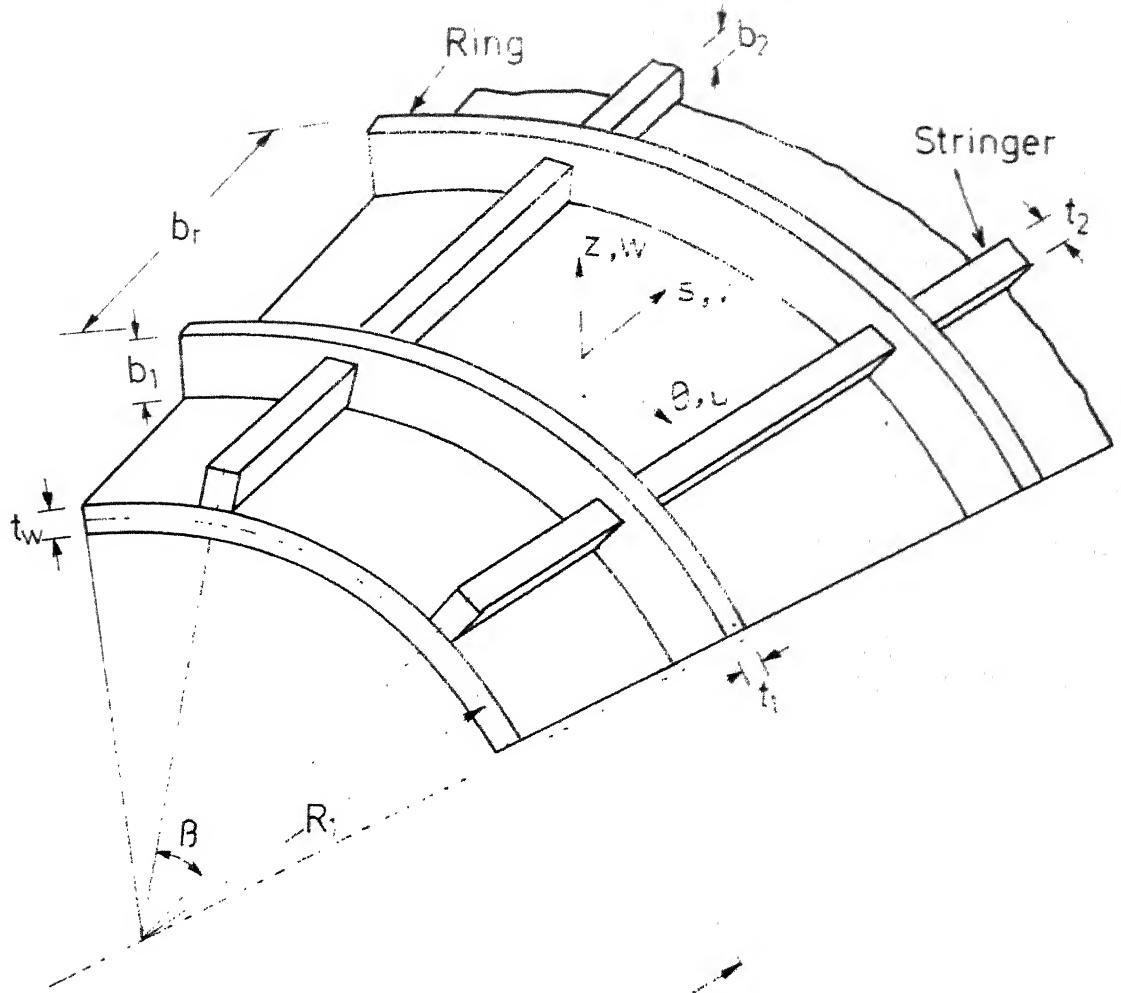
FIG. 3·1 GEOMETRY AND COORDINATE SYSTEM OF CONICAL
SHELL WITH STIFFENRS

The simply supported end conditions are

$$u\,(s_1,\,\theta) = u\,(s_2,\,\theta) = 0$$

and $w\,(s_1,\,\theta) = w\,(s_2,\,\theta) = 0$ $\qquad$ (3.10)

The displacements of the middle surface of the shell, which satisfy the boundary conditions [Eq. (3.10)], are given by

$$u = \bar{u}\,s^2\,\sin\,k_m\,(s - s_1)\,\cos\,n\,\theta$$

$$v = \bar{v}\,s^2\,\cos\,k_m\,(s - s_1)\,\sin\,n\,\theta \qquad (3.11)$$

$$w = \bar{w}\,s^2\,\sin\,k_m\,(s - s_1)\,\sin\,n\,\theta$$

where

$$k_m = \frac{m\pi}{(s_2 - s_1)}$$

$\quad$ m $\;$ is the number of meridional half-waves,

and $\quad$ n $\;$ is the number of circumferential full waves.

The term s^2 is proportional to the cross sectional area enclosed by the shell at a distance s from the cone apex. It is used to shift the point of maximum displacement towards the large end of the cone, a phenomenon that occurs in reality due to decrease in stiffness of the shell with increasing radius [17].

Substituting the solution [Eq. (3.11)] in Eq. (3.9) and applying the condition for minimum potential energy, a matrix equation is obtained as

$$
\begin{bmatrix}
H_1 & H_2 & H_3 \\
H_2 & H_4 & H_5 \\
H_3 & H_5 & H_6 + P_{cr} H_{10}
\end{bmatrix}
\begin{Bmatrix}
\bar{u} \\
\bar{v} \\
\bar{w}
\end{Bmatrix}
=
\begin{Bmatrix}
0 \\
0 \\
0
\end{Bmatrix}
\tag{3.12}
$$

where H_i are defined in Appendix A,

and P_{cr} is the critical load intensity (load per unit
length along the circumference) given by

$$
P_{cr} = [H_6 - \frac{(H_1 H_5^2 + H_3^2 H_4 - 2 H_2 H_3 H_5)}{(H_1 H_4 - H_2^2)}] / H_{10}
\tag{3.13}
$$

3.2 FREE VIBRATION ANALYSIS

The total energy π_T of the stiffened conical
shell, which is in a dynamic state, is the sum of strain
energies of the shell π_c, stringers π_s, rings π_r, and
potential energy due to inertial load (kinetic energy)
π_ω. π_c, π_s and π_r are the same as defined by Eqs.
(3.4), (3.5) and (3.6) respectively. π_ω is given by

$$
\pi_\omega = - \frac{1}{2} \int_{s_1}^{s_2} \int_{o}^{2\pi} [(m_w + m_s + m_r) (\dot{u}^2 + \dot{v}^2 + \dot{w}^2)]
$$

$$
s \cos \alpha \ d\theta \ ds
\tag{3.14}
$$

where

$$
m_w = \rho_w t_w
$$

$$
m_r = \rho_r A_r / b_r
$$

$$
m_s = \rho_s A_s / b_s
$$

$$\therefore \quad \pi_T = \pi_c + \pi_s + \pi_r + \pi_\omega \tag{3.15}$$

The displacement solution for this analysis is assumed as

$$u = \bar{u} \, e^{j\omega t} \, s^2 \sin k_m \, (s - s_1) \cos n\theta$$

$$v = \bar{v} \, e^{j\omega t} \, s^2 \cos k_m \, (s - s_1) \sin n\theta \tag{3.16}$$

$$w = \bar{w} \, e^{j\omega t} \, s^2 \sin k_m \, (s - s_1) \sin n\theta$$

where $k_m = m\pi / (s_2 - s_1)$,

m is the number of meridional half-waves,

and n is the number of circumferential full waves.

Substituting the solution (Eq. 3.16) in the total energy expression (Eq. 3.15), and applying the condition for minimum potential energy, the equations of motion for the vibrating shell are obtained as

$$\begin{bmatrix} (H_1 - \omega^2 H_7) & H_2 & H_3 \\ H_2 & (H_4 - \omega^2 H_8) & H_5 \\ H_3 & H_5 & (H_6 - \omega^2 H_9) \end{bmatrix} \begin{Bmatrix} \bar{u} \\ \bar{v} \\ \bar{w} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \tag{3.17}$$

where H_i are given in Appendix A.

The characteristic equation corresponding to Eq. (3.17) is

$$B_3 Y^3 - B_2 Y^2 + B_1 Y - B_0 = 0 \tag{3.18}$$

where

$$B_3 = 1.0,$$

$$B_2 = -\left(\frac{H_1}{H_7} + \frac{H_4}{H_8} + \frac{H_6}{H_9} \right),$$

$$B_1 = \frac{H_1 H_4 - H_2^2}{H_7 H_8} + \frac{H_1 H_6 - H_3^2}{H_7 H_9} + \frac{H_4 H_6 - H_5^2}{H_8 H_9}$$

$$B_0 = \frac{(H_1 H_4 H_6 + 2 H_2 H_3 H_5 - H_2^2 H_6 - H_3^2 H_4 - H_1 H_5^2)}{H_7 H_8 H_9}$$

and

$$Y = \omega^2 = (2 \pi F)^2$$

Of the three natural frequencies obtained from Eq. (3.18) for each combination of m and n, only the lowest (which is associated with the radial motion of the shell) is of interest.

3.3 STRESS ANALYSIS

For a given axial load (compressive) of intensity P, the resultant stress [Eq. (3.8)] N_s is inversely proportional to s. Hence the maximum occurs at the small end ($s = s_1$) of the shell and the magnitude is given by

$$(N_s)_{max} = \frac{P}{\sin \alpha} \tag{3.19}$$

CHAPTER 4

FORMULATION AND SOLUTION OF THE PROBLEM

This chapter deals with the formulation of optimum
design problems as standard nonlinear programming problems
and also the techniques used for solving these problems.
The design problem of stiffened cylindrical shell is for-
mulated in terms of objective and constraint functions in
Section 4.1. Section 4.2 deals with the formulation of the
problem of design optimization of stiffened conical shell
under buckling, natural frequency and direct stress cons-
traints. The procedure for solving the design problems
is described in Section 4.3.

4.1 FORMULATION OF CYLINDRICAL SHELL PROBLEM

4.1.1 Objective Function

In the design optimization of cylindrical shell,
the objective function normally considered is weight. For
the present problem, the sum of the individual weight of
shell wall W_1 rings, W_2 and stringers W_3 forms the objec-
tive function. Thus the total weight, f, of the shell is
expressed as

$$f = 2 \pi R (t_w \gamma_w a + A_r n_r \gamma_r) + A_s n_s \gamma_s a \qquad (4.1a)$$

where R = mean radius of cylinder,

  t_w = thickness of shell wall,

$$a \quad = \text{ length of shell,}$$

$$\gamma_w, \gamma_r, \gamma_s = \text{ specific weights of shell wall, rings and}$$
$$\text{stringer materials, respectively,}$$

$$A_r, A_s = \text{ cross sectional areas of ring and stringer,}$$
$$\text{respectively,}$$

$$n_r, n_s = \text{ number of rings and stringers, respectively.}$$

4.1.2 Design Variables

In majority of the applications, shells are used as components of a system. For a shell to be compatible with the mating parts, overall dimensions such as mean radius and length of the shell are fixed beforehand. Consequently, the parameters that are taken up as variables for the optimum shell design are shell wall thickness, cross sections of stiffeners, number/spacing of rings and number/spacing of stringers. In the present problem, spacings of stiffeners are treated as continuous variables. The objective function Eq. (4.1a) can be rewritten in terms of spacings of stiffeners as

$$f = 2\pi R\, t_w\, a\, \gamma_w + 2\pi R\, A_r\, \gamma_r\, \frac{a}{l} + A_s\, a\, \gamma_s\, \frac{2\pi R}{d}$$

$$(4.1)$$

where the design variables are given by

$$t_w = \text{ thickness of shell wall}$$

$$t_1 = \text{ thickness of ring}$$

$$b_1 = \text{ depth of ring}$$

$$t_2 = \text{ thickness of stringer}$$

b_2 = depth of stringer

l = spacing of rings

d = spacing of stringers

4.1.3 Behaviour Constraints

Shells, as they are used in space-craft and aero-space industry, have to satisfy certain behaviour constraints like natural frequency, overall buckling strength, local buckling strength and direct stress. These constraints are formulated as follows.

(i) Constraint on Natural Frequency

This is one of the important constraints which must be satisfied to avoid failure by resonance. Resonance occurs when disturbing frequency is equal to one of the natural frequencies of the shell. Hence minimum natural frequency must be constrained to be not less than the disturbing ones.

Theoretically, the shell has infinite natural frequencies, which are given by Eq. (2.11) for different integer values of m greater than or equal to one and n greater than or equal to zero. In practice the minimum of these, which occurs for finite values of m and n, are constrained. As the combination of m and n which gives minimum frequency is not known beforehand, frequencies have to be computed for all combinations of m and n

until minimum is obtained. For verification, finite upper bounds for m and n are assumed and natural frequencies are computed for a number of shells with different stiffeners. From these results, it is observed that minimum frequency in each case occurs always at m equal to one only. Hence m can be taken as one and n has to be varied to find the minimum frequency. In an optimization routine, this minimum has to be found a number of times, which requires a lot of computational time. Hence the following procedure is adopted.

A design vector of independent variables is assumed and the frequency spectrum is obtained at the initial design configuration of the shell. In this spectrum let the value of n corresponding to minimum frequency be n'. As the design vector changes in optimization routine, the value n which gives minimum frequency will also change but it is expected to be in the neighbourhood of the starting value n'. Hence frequencies at the values of n equal to n' - 1, n' and n' + 1 are constrained as

$$F_i \geqslant F_{min} \qquad i = 1, 2, 3 \tag{4.2}$$

where F_i are taken as $F_1 = F$ (m = 1, n = n' - 1), $F_2 = F$ (m = 1, n = n') and $F_3 = F$ (m = 1, n = n' + 1), and F_{min} is lower bound on frequency.

(ii) Constraint on Buckling Strength

In practice, shells are subjected to different
combinations of axial, external and internal loads. These
loads sometimes lead to the failure of the shell due to
buckling before the induced stress reaches the yield
stress. Hence, a constraint on the buckling strength is
necessary to overcome this failure.

A procedure similar to the case of natural fre-
quencies is adopted in forming the constraints on the
buckling strength, except that m is not kept constant in
this case. If the values of m and n corresponding to
the minimum buckling load are m" and n" at the starting
design vector, buckling loads at different combinations of
m⁻ (viz., m" - 1, m" and m" + 1) and n (viz., n" - 1, n
and n" + 1) are constrained as

$$N_{x_i} \; 2 \pi R \; \bar{t} \geq P \quad \text{applied x Factor of safety.} \bar{t}$$
$$i = 1, 2, \ldots 9 \qquad\qquad (4.3)$$

where N_{x_i} are given by Eq. (2.9) with $N_y = 0$

and $\bar{t} = t_w + \dfrac{A_s}{d}$.

(iii) Constraint on Local Buckling Strength:

Besides the constraints on frequency and overall
buckling strength, it is safer to have constraints on
local buckling strength also. This need arises due to

local imperfections and inaccuracies in fabrication.
Failure may occur because of local buckling of the shell
wall or buckling of stiffeners.

Considering the panel between any two stringers,
the expression for critical buckling stress is given by
Eq. (2.13). The constraint on this critical stress is
expressed as

$$\sigma cr_w \geq (P_{applied} \times Factor\ of\ safety/2\pi R\ \bar{t}) \qquad (4.4a)$$

Similarly the constraint on critical buckling
stress in rectangular stringers given by Eq. (2.14), can
be expressed as

$$\sigma cr_s \geq (P_{applied} \times Factor\ of\ safety/2\pi R\ \bar{t}) \qquad (4.4b)$$

In addition to the above, the stringer also may
fail due to column buckling. The constraint on the buck-
ling load Eq. (2.15) can be expressed as

$$P_{cr_s} \geq (P_{applied} \cdot \frac{A_s}{2\pi R\ \bar{t}} \cdot Factor\ of\ safety) \qquad (4.4c)$$

(iv) Constraint on Yield Stress

The induced stress in the shell wall is given
by Eq. (2.12) should be less than the allowable stress
of the material, i.e.,

$$(P_{applied}/ 2\pi R\ \bar{t}) \leq (Yield\ stress/Factor\ of\ safety) \qquad (4.5)$$

Besides the above constraints, each design variable is constrained by a lower bound.

4.2 FORMULATION OF CONICAL SHELL PROBLEM

4.2.1 Objective Function

In the design of conical shell also, the total weight of the shell is taken as the objective function for minimization. The total weight, f, which is the sum of weights of skin (W_1), rings (W_2) and stringers (W_3) is given by

$$f = W_1 + W_2 + W_3 \qquad (4.6)$$

where

$$W_1 = \pi (R_2 s_2 - R_1 s_1)\, t_w \gamma_w$$

$$W_2 = 2\pi \left[\sum_{i=1}^{n_r} \left\{ R_1 + (i - 0.5) b_r \, \sin(90-\alpha) \right\} \right] A_r \gamma_r$$

and $W_3 = (s_2 - s_1)\, n_s A_s \gamma_s$

4.2.2 Design Variables

Shell thickness, cross sections of stiffeners, number of rings and number of stringers are considered as variables for the optimum design of conical shell. Similar to the case of cylindrical shell, the radii of the conical shell and the angle α (Fig.3.1) are assumed to be fixed

beforehand.   The design variables are

t_w =   thickness of shell wall

t_1 =   thickness of ring

b_1 =   depth of ring

t_2 =   thickness of stringer

b_2 =   depth of stringer

n_r =   number of rings

n_s =   number of stringers.

4.2.3   Behaviour Constraints

In the case of conical shell constraints on frequency, buckling and yield stress are imposed as in the case of cylindrical shell.   These constraints can be stated as follows.

(i)   Constraint on Natural Frequency

Unlike in the case of cylindrical shell, inplane inertias are considered in the vibration analysis of conical shell.   Hence for a given set of values   m   and   n,   there exists three natural frequencies.   Of these three, the lowest one which corresponds to the radial motion of the shell is of interest for any set of values   m   and   n .   The minimum of such lowest ones should not be less than a predetermined constant.   To reduce the computational time required in finding the minimum frequency, the following procedure is adopted.

It is observed that minimum frequency occurs for
m equal to one, and hence m is set equal to one. The
value of n which traps the minimum is supposed to be
finite. This finite value is taken as six which is the
case with the starting design vector for the example prob-
lem considered. The resulting constraints are expressed as

$$F_i \geq F_{min} \qquad i = 1, 2, \ldots 7 \qquad (4.7)$$

where $F_i = F (m = 1, n = i - 1)$, $i = 1$ to 7 and F is
defined by Eq. (3.18).

(ii) Constraint on Buckling Strength

Conical shells, subjected to axial loads and or /
circumferential loads or both, may fail due to lack of
buckling strength. Hence, buckling strength should be more
than a predetermined value. This constraint is formulated
as described below.

Buckling load, as expressed by Eq. (3.13), is a
function of the axial and circumferential mode numbers m
and n. Hence, a procedure similar to the one described for
Eq. (4.3) is adopted, to express this constraint as:

$$P_{cr_i} \geq P_{applied} \qquad i = 1, 2, \ldots 9 \qquad (4.8)$$

where P_{cr_i} are defined by Eq. (3.13)
and $P_{applied}$ is the applied load intensity.

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(iii) Constraint on Yield Stress

To avoid failure of the material of shell wall by yielding, the induced stress in the shell given by Eq. (3.19) should be less than the allowable stress, thus

$$\frac{P_{applied}}{t \ \sin\alpha} \leqslant \frac{\text{Yield stress}}{\text{Factor of safety}} \tag{4.9}$$

In addition to the above behaviour constraints, each design variable is constrained by a lower bound value.

4.3 SOLUTION PROCEDURE

The design optimization problems formulated in Sections 4.1 and 4.2 can be stated as standard nonlinear programming problems as follows:

Find $\vec{X}$ which minimizes the objective function $f(\vec{x})$, subject to the constraints

$$g_j(\vec{X}) \leqslant 0 \qquad j = 1, 2, \ldots p \tag{4.10}$$

where $\vec{X}$ is the vector of design variables.

These problems are solved by gradient method (Davidon - Fltecher - Powell method) with interior penalty function formulation. For one dimensional minimization, cubic interpolation method is used [44]. As the function to be minimized is not in a simple and explicit form, finite differences (backward differences) technique is used for calculation of the gradient. Since the

Kuhn - Tucker conditions are necessarily to be satisfied at the optimum point, the Lagrange multipliers ($\vec{\lambda}$) are obtained from the gradients of active constraints (g_j, $j = 1, \ldots p'$) and the objective function (f) [44] as

$$\vec{\lambda} = \left\{ [G]^T [G] \right\}^{-1} [G]^T \vec{F} \tag{4.11}$$

where

$$[G] = \begin{bmatrix} \dfrac{\partial g_1}{\partial x_1} & \dfrac{\partial g_2}{\partial x_1} & \cdots & \dfrac{\partial g_p}{\partial x_1} \\ \dfrac{\partial g_2}{\partial x_2} & & & \\ \vdots & & & \\ \dfrac{\partial g_1}{\partial x_q} & \dfrac{\partial g_2}{\partial x_q} & \cdots & \dfrac{\partial g_p}{\partial x_q} \end{bmatrix}$$

$$\vec{\lambda} = \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_{p'} \end{Bmatrix} \quad , \quad \vec{F} = \begin{Bmatrix} -\dfrac{\partial f}{\partial x_1} \\ -\dfrac{\partial f}{\partial x_2} \\ -\dfrac{\partial f}{\partial x_q} \end{Bmatrix}$$

and q is number of variables.

The condition for any design vector to be a local optimum, the Lagrange multipliers computed at that vector Eq. (4.11) should be positive.

CHAPTER 5

RESULTS AND DISCUSSION

The optimization problems formulated in chapter four are solved and the numerical results are presented in this chapter.

5.1 CYLINDRICAL SHELL

With a view of studying the effects of different sets of constraints and also the influence of type of stiffeners on the optimum solution, several cylindrical shell optimization problems are solved. The different problems considered are summarized in Table 5.1.

TABLE 5.1

Types of stiffeners used		Problems solved		
for stringers	for rings	with frequency and side constraints	with overall buckling and side constraints	With all constraints
Rectangular	Rectangular	✓	✓	✓
Z-section	Tubular	✓	✓	✓
Z-section	I-section	✓	✓	✓

The data used in all these problems is given in Table 5.2. The numerical results are shown in Table 5.3 to 5.5. The computer time taken for the solution of

an optimization problem ranged from 4-11 minutes on
IBM 7044 computer.

5.1.1 Effect of Constraints

For studying the influence of different sets of
constraints on the problem, the same starting point and
side constraints are used for any particular combination
of stiffeners.

The following observations can be made from the
numerical results obtained:

(a) When the behavioural constraint set consists of only
natural frequency restriction, the frequency cons-
traint has become active for all combinations of
stiffeners. Further in all these cases, the spacings
of rings and stringers approached their respective
upper bounds, while the thicknesses of skin, ring and
stringer and depth of stringer have reached their
respective lower bounds at the final solution.

(b) When overall buckling is considered as the only beha-
viour constraint, it has been found to be critical
at the optimum point for any set of stiffeners.
Here also the spacings of rings and stringers approa-
ched their upper bound values and the shell thickness
reached its lower limit. For all types of stiffeners
all the side constraints except the lower bound on
the depth of stringer are active.

(c) When all the constraints are considered the constraint
      on natural frequency, panel buckling strength and
      direct stress have become active at the optimum point
      in all the cases. The skin thickness has not reached
      its lower limit in any of the cases.

      In the case of rectangular ring - rectangular
stringer combination, side constraint on the stringer
thickness as well as stringer buckling (as column) cons-
traint are active. For Z - stringer-tubular ring combina-
tion the thicknesses of rings and stringers have reached
their respective lower bounds at the optimum point. For
the combination of Z - stringers and I - rings, the
lower bound on ring thickness and upper bound on ring
spacing have become critical at the optimal solution.

      The weight of the stiffened shell has been found
to have a higher value, when all the constraints are con-
sidered. This is to be expected as the feasible design
space becomes narrower with the addition of constraints.
The stringers are found to contribute more to the total
weight of the shell than that of rings, with allcons-
traints.

5.1.2 Effect of Type of Stiffeners

      As stated earlier, the following stiffener
combinations are considered for studying the effect of

type of stiffeners on the optimization problem:

Case 1:   rectangular stringers and rectangular rings,

Case 2:   Z - section stringers and tubular rings,

Case 3:   Z - section stringers and I - section rings.

In order to have a meaningful comparison of the results, the ring cross sectional area and stringer cross sectional area are taken to be same in all the three cases at the initial design vector. The lower bounds on thicknesses and depths are also taken such that the stiffener areas are constrained by the same lower bound in all the cases.

It can be seen that, for all sets of constraints considered, I - ring combination (Case 3) gives the minimum weight. When the behaviour constraint is only on natural frequency, the minimum weight obtained in cases 1, 2 and 3 are 79.08, 85.36 and 73.12 Kg. respectively. Thus the minimum weights corresponding to cases 1 and 2 are 8.15% and 16.75 % higher than that of case 3, respectively.

When overall buckling only is considered, the minimum weights corresponding to cases 1, 2 and 3 are 78.34, 76.81, 76.27 Kg. respectively. Thus cases 1 and 2 led to 2.72 % and 0.71 % higher weights, respectively compared to case 3.

When all the constraints are considered, the minimum weights obtained are 189.95, 187.61 and 177.77 kg in cases 1, 2 and 3 respectively. This means that the minimum weights in cases 1 and 2 are higher by 6.86 % and 5.54% respectively compared to case 3.

5.1.3 Comparison with the Results Reported in Ref. [26]

Morrow and Schmit presented the optimization results of a cylindrical shell whose data is given in Table 5.6, by considering minimization of weight as objective with overall buckling and local buckling as constraints. In this reference the buckling load has been obtained by solving the equilibrium equations.

The same problem is solved with constraints on natural frequency, overall buckling, local buckling, direct stress and side constraints.

The starting design vector is taken to be same as that of Ref. [26], and the results of optimization are given in Table 5.7. It can be seen that the present constraint set resulted in an increase of 31.22% in the minimum weight. A comparison of the present design vector with that of Ref. [26] shows that the two design vectors (especially the thicknesses of shell) are different at the optimal solution. This, along with the results of Sections 5.1.1 and 5.1.2, shows that the shell thickness

tries to reach its lower bound value at the optimum point
when one behaviour constraint only is considered in the
problem. However, when all the behaviour constraints are
considered the shell thickness will be in general different
from its bound value.

5.2 CONICAL SHELL

A conical shell with rectangular stiffeners,
having the same overall dimensions as the viking shell [40]
is considered for the optimum design. The optimum design
of the conical shell is found under three different cons-
traint sets. The first constraint set consists restric-
tions on natural frequency and design variables. The
second one contains overall buckling and side constraints
on design vector. In the third set, constraints are placed
on natural frequency, overall buckling strength, direct
stress induced and on the design variables.

The data considered for numerical computation is
shown in Table 5.8. The results of optimization are shown
in Table 5.9. The computer time taken for the solution of
one problem ranged from 7 to 12 minutes on IBM 7044
computer.

It can be seen that these results show a similar
behaviour as in the case of cylindrical shells. Some of
the characteristics are as follows:

(i)   the minimum weight obtained in the case of third
        constraint set is higher than that of the first
        two cases where only one behaviour constraint
        was considered.

(ii)   the skin thickness approached its lower bound
        value in the first two cases and not in the third
        case,

(iii)   the respective behaviour constraint has become
        active at the optimum point in the first two
        cases, while the constraint on direct stress has
        been found to be critical in the third case.

In this problem, the number of rings and stringers
are taken as design variables instead of stiffener spacings.
It can be seen that the number of rings reached its minimum
permissible value in the first case only, whereas the number
of stringers has not reached its lower bound in any of the
three cases.

A few optimum solutions of cylindrical and conical
shell examples are considered for checking the satisfaction of
Kuhn-Tucker conditions and the Lagrange multipliers obtained
for the solutions are found to be positive. This shows that
the solutions obtained are atleast local minima.

TABLE 5.2

Numerical Data for Cylindrical Shell

Material of the shell, stringers and rings	:	Aluminium
Length of the cylinder (a)	:	243.84 cm
Mean radius of cylinder (R)	:	121.92 cm
Axial load	:	3.63×10^5 kg
Yield stress of the material	:	2100 kg/cm^2
Factor of safety for the yield stress	:	1.5
Factor of safety for the buckling load	:	2.0
Lower bound on natural frequency	:	70.0 c/s
Young's modulus of material (E)	:	7.35×10^5 kg/cm^2
Poissons ratio (ν)	:	0.32
Specific weight of the material (γ)	:	2.713×10^{-3} kg/cm^3
Acceleration due to gravity (g)	:	9.81 m/sec^2
Position of stringer	:	Outside
Position of rings	:	Outside

For Z - section web is perpendicular to flange and web to
flange ratio is 2.5.

For I - section top and bottom flanges are equal. Web
to flange ration is 2.0.

TABLE 5.3

Variable/ objective function	Bounds	Starting design vector	Optimum design vector		
			with frequency constraint	with overall buckling constraint	with all constraints
t_w	0.1270	0.5080	0.1275	0.1270	0.3033
t_1	0.3810	1.2700	0.3929	0.3840	0.4183
b_1	0.7620	1.2700	1.4765	0.7661	1.5138
t_2	0.3810	1.2700	0.4135	0.3830	0.3866
b_2	0.7620	1.2700	0.7798	2.9726	1.2997
l	24.3840 (upper)	8.1280	24.3590	24.0280	19.0250
d	76.6060 (upper)	25.5520	74.6760	75.9460	13.3860
f	–	390.7600	79.0800	78.3400	189.9500
W_1	–	257.9500	64.8000	64.5200	153.9600
W_2	–	100.7600	12.0900	6.2100	16.9100
W_3	–	32.0500	2.1900	7.6100	19.0700
Frequency (c/s)		78.5807	70.1584	–	70.2400
Buckling strength (Kg)		1.86×10^6	–	7.31×10^5	8.5×10^5
Panel buckling stress (kg/cm^2)	critical	1.065×10^3			1.385×10^3
	induced	8.26×10^2			1.385×10^3
Stringer buckling (as column) load kg	critical	1.20×10^4			7.14×10^2
	induced	1.34×10^3			7.00×10^2
Stringer buckling (as plate) stress (kg/cm^2)	critical	33.67×10^4			2.98×10^4
	induced	8.26×10^2			1.38×10^3
Direct stress (kg/cm^2)		8.26×10^2			1.38×10^3

TABLE 5.4

Variable/ objective function	Bounds	Starting design vector	Optimum design vector with frequency cons-traint	with over-all buck-ling cons-traint	With all cons-traints
t_w	0.1270	0.5080	0.1273	0.1283	0.2187
t_1	0.1016	0.2540	0.1016	0.1052	0.1017
b_1	0.5080	1.1430	1.4402	0.5108	1.2256
t_2	0.1016	0.2540	0.1016	0.1052	0.1184
b_2	1.7272	3.8100	1.7399	4.3099	5.4491
l	24.3840 (upper)	8.1280	24.1050	24.3080	23.0890
d	76.6060 (upper)	25.5520	71.0440	75.5140	9.5760
f		390.7600	85.3600	76.8100	18.7610
W_1		257.9500	64.5600	65.1600	111.0500
W_2		100.7600	18.6700	6.3200	16.5000
W_3		32.0500	2.1300	5.3300	60.0600
Frequency c/s		75.0100	70.6800	-	71.0000
Buckling strength (kg)		4.04×10^6	-	7.28×10^5	7.47×10^6
Panel buck-ling stress (kg/cm^2)	critical	1064.7500	-	-	1405.0900
	induced	825.8100	-	-	1399.4699
Stringer buckling (as column) load (Kg)	critical	17.76×10^4	-	-	3.46×10^4
	induced	1.34×10^3	-	-	1.600×10^3
Direct stress (kg/cm^2)		825.8100	-	-	1.394×10^3

TABLE 5.5

Variable/ objective function	Bounds	Starting design vector	Optimum design vector		
			with frequency constraint	with overall buckling constraint	With all constraints
t_w	0.1270	0.5080	0.1270	0.1273	0.2235
t_1	0.1016	0.2540	0.1016	0.1019	0.1054
b_1	1.5304	3.4290	1.6203	1.5578	1.5692
t_2	0.1016	0.2540	0.1029	0.1039	0.1359
b_2	1.7272	3.8100	1.7871	4.2172	4.6703
l	24.3840 (upper)	8.1280	24.1810	23.8760	23.4950
d	76.6060 (upper)	25.5520	74.3460	72.5930	9.7540
f	–	390.7600	73.1200	76.2700	177.7700
W_1	–	257.9500	64.5200	64.6100	113.4900
W_2	–	100.7600	6.4900	6.3000	6.6800
W_3	–	32.0500	2.1100	5.3600	57.6000
Frequency c/s		121.5400	70.6700	–	74.5200
Buckling strength (kg)		4.45×10^6	–	7.77×10^5	4.41×10^6
Panel buckling stress (kg/cm^2) : critical		1064.7500	–	–	1414.7700
Panel buckling stress (kg/cm^2) : induced		825.79	–	–	1400.0000
Stringer buckling (as column) load (kg) : critical		177210.13	–	–	23591.7600
Stringer buckling (as column) load (kg) : induced		1337.76	–	–	1554.4800
Direct stress (kg/cm^2)		825.7900	–	–	1400.0000

TABLE 5.6

Numerical Data for the Example From Ref. [26]

Length of cylinder (a)	: 96.5 cm
Mean radius of cylinder (R)	: 24.2 cm
Yield stress of the material	: 3.5×10^3 kg/cm^2
Poissons ratio	: 0.33
Lower bound on natural frequency	: 100 c/s
Young's modulus of material (E)	: 7.35×10^5 kg/cm^2
Axial load per unit length of circumference	: 143.2 kg/cm
Specific weight of material (γ)	: 0.00278 kg/cm^3
Factor of safety for direct stress	: 1.5
Factor of safety for buckling strength	: 2.0
Placing of stiffeners	: Outside
Type of stiffeners	: Rectangular

TABLE 5.7

Variable/ objective function	Bounds	Starting design vector	Optimum design vector	
			with present constraint set	As given in Ref. [26]
t_w	0.0516*	0.0711	0.0442	0.02360
t_1	0.0254*	0.1270	0.0254	0.05160
b_1	0.1270	0.2540	0.1748	4.64800
t_2	0.0254*	0.1270	0.0488	0.03810
b_2	0.1270	0.2540	0.5286	0.36070
l	12.7000 (upper)	3.8100	6.0096	6.98500
d	5.0800 (upper)	0.6350	1.5037	0.67060
f	-	5.3600	2.5589	1.95000
W_1	-	2.9300	1.8222	0.96500
W_2	-	0.3500	0.03157	0.97600
W_3	-	2.0860	0.70520	0.00900

* These bounds in the original Ref [26] are 0.0.

TABLE 5.8

Numerical Data for Conical Shell

Material of shell, stringers and rings	: Magnesium
Mean radius of cone at the small end (R_1)	: 81.28 cm
Mean radius of cone at the base (R_2)	: 175.26 cm
Cone apex angle ($90 - \alpha$)	: 70.0°
Axial load	: 4.536×10^4 kg
Yield stress of the material	: 1120 kg/cm^2
Factor of safety for yield stress	: 1.5
Factor of safety for buckling	: 2.0
Lower bound on frequency	: 75.0 c/s
Youngs modulus of material (E)	: 4.06×10^5 kg/cm^2
Poissons ratio (μ)	: 0.33
Specific weight of material (γ)	: 1.8279×10^{-3} kg/cm^3
Position of stringer	: Outside
Position of rings	: Outside

TABLE 5.9

Variable/ Objective function	Lower bounds	Starting design vector	Optimum design vector		
			With frequency constraint	With overall buckling constraint	With all constraints
t_w (cm)	0.1270	0.5080	0.1280	0.1331	0.1839
t_1 (cm)	0.3810	1.2700	0.3924	0.4013	0.4648
b_1 (cm)	0.7620	1.2700	0.7628	0.7874	0.7945
t_2 (cm)	0.3810	1.2700	0.3820	0.5004	0.6761
b_2 (cm)	0.7620	1.2700	0.7884	3.3421	5.8473
n_r	10	30	12	23	28
n_s	10	30	18	17	21
f (kg)	-	155.2100	25.1730	35.5705	57.5775
W_1 (kg)	-	74.9600	18.8800	19.6431	27.1194
W_2 (kg)	-	71.3900	5.3000	10.7224	15.2555
W_3 (kg)	-	8.8600	0.9930	5.2050	15.2026
Frequency c/s	104.4800	79.9100	-		199.1500
Buckling strength (kg)	3.17×10^5	-		1.95×10^5	11.74×10^5
Direct stress (kg/cm^2)	6.13×10^3	-		-	10.66×10^3

CHAPTER 6

CONCLUSIONS AND RECOMMENDATIONS

6.1 CONCLUSIONS

The following conclusions can be drawn from the results obtained for the optimization of cylindrical and conical shells:

i) When the natural frequency or overall buckling strength is the only behaviour constraint considered, it is becoming active at the optimum point.

ii) The objective function (weight of the shell) is found to be very sensitive to the design variable, t_w. The side constraint on t_w is becoming active when only one behaviour constraint and the side constraints are considered.

iii) In most of the cases the side constraints on thicknesses of ring and stringer are becoming active.

iv) Out of the three combinations of stiffeners considered, the Z - section stringers and I - section rings combination is found to give minimum weight.

v) In the case of conical shell, the number of rings is coming to the lower bound under any constraint set.

vi) The optimum weights of the cylindrical as well as conical shells are found to increase as the number of constraints considered are increased.

vii) The stringers are seen to contribute considerably to the total weight of the shell, when all the constraints are considered.

6.2 RECOMMENDATIONS FOR FUTURE WORK

i) The procedure adopted in forming the natural frequency and overall buckling constraint is an approximate one. It can be made more accurate by searching for the minimum frequency and buckling strength in the (m, n) space at each design vector in the optimization routine.

ii) The influence of inside stiffeners or combination of inside and outside stiffeners (rings inside - stringers outside or stringers inside - rings outside) on the optimum design can also be studied.

iii) In order to study the effect of type of stiffeners on the design of conical shells, a number of

combinations of stiffeners are to be considered in addition to the rectangular stiffeners considered in this work.

iv) The design optimization can be carried with other criteria like maximization of minimum natural frequency and maximization of the difference between the lowest two natural frequencies.

v) The mixed-integer nonlinear programming techniques can be used to solve the problem with integer variables (namely, number of rings and number of stringers) more efficiently.

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Expressions for H_1, H_2, ..., H_{10} of Eqs. (3.12) and (3.17)

$$H_1 = D_w \frac{\pi}{\cos\alpha} (IN8)\, n^2 + D_r \frac{\pi}{\cos\alpha}\, n^2 + \frac{(1 - \mu_w)}{2} \, .$$

$$D_w \, \pi \cos\alpha \left[(IN8) + 2\, k_m\, (IN\ 14) + k_m^2\, (IN\ 4) \right]$$

$$+ S_r \frac{2\pi \sin\alpha}{\cos^2\alpha} (IN\ 7)\, n^2 + K_w \frac{\pi\, \sin^2\alpha}{\cos^3\alpha} (IN\ 6)\, n^2$$

$$+ K_r \frac{\pi \sin^2\alpha}{\cos^3\alpha} (IN\ 6)\, n^2 + \frac{(1 - \mu_w)}{2} K_w \frac{\pi k_m^2\, \sin^2\alpha}{\cos\alpha} (IN\ 2)$$

$$+ T_s \frac{k_m^2\, \sin^2\alpha}{\cos\alpha} (IN\ 1)\, (IN\ 16)$$

$$H_2 = - D_w\, \pi\, (IN\ 13)\, n - D_r\, \pi\, (IN\ 13)\, n - \mu_w\, D_w\, \pi \left[2\, (IN\ 13) \right.$$

$$\left. - k_m\, (IN\ 9) \right] n + \frac{(1 - \mu_w)}{2} D_w\, \pi \left[(IN\ 13) + k_m\, (IN\ 3) \right] n$$

$$- S_r \frac{\pi\, \sin\alpha}{\cos\alpha} (IN\ 12)\, n$$

$$H_3 = - D_w \frac{\pi\, \sin\alpha}{\cos\alpha} (IN\ 8)\, n - D_r \frac{\pi\, \sin\alpha}{\cos\alpha} (IN\ 8)\, n$$

$$+ S_r\ \pi \left[(IN\ 7)\ \left(2 - \frac{n^2 + \sin^2\alpha}{\cos^2\alpha} \right) + k_m\, (IN\ 13) \right] n$$

$$+ K_w \frac{\pi \sin\alpha}{\cos\alpha} \left[(IN\ 6)\ \left(2 - \frac{n^2}{\cos^2\alpha} \right) + k_m\, (IN\ 12) \right] n$$

$$+ K_r \frac{\pi \sin\alpha}{\cos\alpha} \left[(IN\ 6)\ \left(2 - \frac{n^2}{\cos^2\alpha} \right) \right.$$

$$+ \; k_m \; (IN \; 12) \Big] \, n + \mu_w \; K_w \; \frac{\pi \sin\alpha}{\cos\alpha} \Big[2 \; (IN \; 6)$$

$$+ \; 4 \; k_m \; (IN \; 12) - k_m^2 \; (IN \; 8) \Big] \, n$$

$$- \; \frac{(1 - \mu_w)}{2} \; K_w \; \frac{2 \pi \; k_m \; \sin\alpha}{\cos\alpha} \; \Big[(IN \; 12) + k_m \; (IN \; 2) \Big] \, n$$

$$- \; T_s \; \frac{k_m^2 \; \sin\alpha}{\cos\alpha} \; (IN \; 1) \; (IN \; 16) \; n$$

$$H_4 \; = \; D_w \; \pi \; \cos\alpha \; (IN \; 2) + D_r \; \pi \; \cos\alpha \; (IN \; 2)$$

$$+ \; D_w \; \pi \; \cos\alpha \; \Big[3 \; (IN \; 2) - 4 \; k_m \; (IN \; 14) + k_m^2 \; (IN \; 10) \Big]$$

$$+ \; D_s \; \cos\alpha \; \Big[4 \; (IN \; 1) - 4 \; k_m \; (IN \; 13) + k_m^2 \; (IN \; 9) \Big] \; (IN \; 16)$$

$$+ \; \mu_w \; D_w \; 2 \; \pi \; \cos\alpha \; \Big[2 \; (IN \; 2) - k_m \; (IN \; 14) \Big]$$

$$+ \; \frac{(1 - \mu_w)}{2} \; D_w \; \frac{\pi}{\cos\alpha} \; (IN \; 2) \; n^2$$

$$H_5 \; = \; D_w \; \pi \; \sin\alpha \; (IN \; 13) + D_r \; \pi \; \sin\alpha \; (IN \; 13)$$

$$+ \; \mu_w \; D_w \; \pi \; \sin\alpha \; \Big[2 \; (IN \; 13) - k_m \; (IN \; 9) \Big]$$

$$- \; S_r \; \pi \; \cos\alpha \; \Big[(IN \; 12) \; (2 - \frac{n^2}{\cos^2\alpha} \;) + k_m \; (IN \; 2) \Big]$$

$$- \; S_s \; \cos\alpha \; \Big\{ 4 \; (IN \; 11) + k_m \; \Big[8 \; (IN \; 1) - 2 \; (IN \; 7) \Big]$$

$$- \; 6 \; k_m^2 \; (IN \; 13) + k_m^3 \; (IN \; 9) \Big\} \; (IN \; 15)$$

$$H_6 = D_w \frac{\pi \sin^2 \alpha}{\cos \alpha} (IN\ 8) + D_r \frac{\pi \sin^2 \alpha}{\cos \alpha} (IN\ 8)$$

$$- S_r\ 2\ \pi \sin \alpha \left[(IN\ 7)\ (2 - \frac{n^2}{\cos^2 \alpha}) + k_m\ (IN\ 13) \right]$$

$$+ K_w\ \pi \cos \alpha \left[(IN\ 6)\ (2 - \frac{n^2}{\cos^2 \alpha})^2 \right.$$

$$\left. + 2\ k_m\ (IN\ 12)\ (2 - \frac{n^2}{\cos^2 \alpha}) + k_m^2\ (IN\ 2) \right]$$

$$+ K_r\ \pi \cos \alpha \left[(IN\ 6)\ (2 - \frac{n^2}{\cos^2 \alpha})^2 + 2\ k_m\ (IN\ 12) \right.$$

$$\left. (2 - \frac{n^2}{\cos^2 \alpha}) + k_m^2\ (IN\ 2) \right] + K_w\ \pi \cos \alpha \left\{ 4\ (IN\ 6) \right.$$

$$+ 16\ k_m\ (IN\ 12) - 4\ k_m^2 \left[(IN\ 8) - 4\ (IN\ 2) \right]$$

$$- 8\ k_m^3\ (IN\ 14) + k_m^4\ (IN\ 10) \left. \right\} + K_s \cos \alpha \left\{ 4\ (IN\ 5) \right.$$

$$+ 16\ k_m\ (IN\ 11) - 4\ k_m^2 \left[(IN\ 7) - 4\ (IN\ 1) \right] - 8\ k_m^3\ (IN\ 13)$$

$$+ k_m^4\ (IN\ 9) \left. \right\} (IN\ 15) + \mu_w\ K_w\ 2 \pi \cos \alpha \left\{ \left[2(IN\ 6) \right. \right.$$

$$+ 4\ k_m\ (IN\ 12) - k_m^2\ (IN\ 8) \left. \right]\ (2 - \frac{n^2}{\cos^2 \alpha})$$

$$+ 2\ k_m\ (IN\ 12) + 4\ k_m^2\ (IN\ 2) - k_m^3\ (IN\ 14) \left. \right\}$$

$$+ \frac{(1 - \mu_w)}{2}\ K_w\ \frac{4\ \pi}{\cos \alpha} \left[(IN\ 6) + 2\ k_m\ (IN\ 12) \right.$$

$$+ k_m^2\ (IN\ 2) \left. \right]\ n^2 + T_r\ \frac{\pi}{\cos \alpha} \left[4\ (IN\ 6) \right.$$

$$+ 4\ k_m\ (IN\ 12) + k_m^2\ (IN\ 2) \left. \right]\ n^2 + T_s\ \frac{k_m^2}{\cos \alpha}\ (IN\ 1)\ .$$

$$(IN\ 16)\ n^2$$

$$H_7 = m_w \, \pi \cos\alpha \, (\text{IN } 10) + m_r \, \pi \cos\alpha \, (\text{IN } 10)$$
$$+ \, m_s \cos\alpha \, (\text{IN } 9) \, (\text{IN } 16)$$

$$H_8 = m_w \, \pi \cos\alpha \, (\text{IN } 4) + m_r \, \pi \cos\alpha \, (\text{IN } 4)$$
$$+ \, m_s \cos\alpha \, (\text{IN } 3) \, (\text{IN } 15)$$

$$H_9 = m_w \, \pi \cos\alpha \, (\text{IN } 10) + m_r \, \pi \cos\alpha \, (\text{IN } 10)$$
$$+ \, m_s \cos\alpha \, (\text{IN } 9) \, (\text{IN } 15)$$

$$H_{10} = \pi \, S_1 \left[k_m^2 \, (\text{IN } 3) + 4 \, (\text{IN } 7) + 4 \, k_m \, (\text{IN } 13) \right] \cot\alpha$$

where

$$(\text{IN } 1) = \tfrac{1}{6} \, (S_2^3 - S_1^3) + (1/4 \, k_m^2) \, (S_2 - S_1)$$

$$(\text{IN } 2) = \tfrac{1}{8} \, (S_2^4 - S_1^4) + (3/8 \, k_m^2) \, (S_2^2 - S_1^2)$$

$$(\text{IN } 3) = \tfrac{1}{10} \, (S_2^5 - S_1^5) + (1/2 \, k_m^2) \, (S_2^3 - S_1^3) - (3/4 \, k_m^4) \, (S_2 - S_1)$$

$$(\text{IN } 4) = \tfrac{1}{12} \, (S_2^6 - S_1^6) + (5/8 \, k_m^2) \, (S_2^4 - S_1^4) - (15/8 \, k_m^4)(S_2^2 - S_1^2)$$

$$(\text{IN } 5) = \tfrac{1}{2} \, (S_2 - S_1)$$

$$(\text{IN } 6) = \tfrac{1}{4} \, (S_2^2 - S_1^2)$$

$$(\text{IN } 7) = \tfrac{1}{6} \, (S_2^3 - S_1^3) - (1/4 \, k_m^2) \, (S_2 - S_1)$$

$$(\text{IN } 8) = \tfrac{1}{8} \, (S_2^4 - S_1^4) - (3/8 \, k_m^2) \, (S_2^2 - S_1^2)$$

$$(\text{IN } 9) = \tfrac{1}{10} \, (S_2^5 - S_1^5) - (1/2 \, k_m^2) \, (S_2^3 - S_1^3) + (3/4 \, k_m^4)(S_2 - S_1)$$

$$(\text{IN } 10) = \tfrac{1}{12} \, (S_2^6 - S_1^6) - (5/8 \, k_m^2) \, (S_2^4 - S_1^4) + (15/8 \, k_m^4)(S_2^2 - S_1^2)$$

(IN 11) $= -(1/4 \ k_m) (s_2 - s_1)$

(IN 12) $= -(1/4 \ k_m) (s_2^2 - s_1^2)$

(IN 13) $= -(1/4 \ k_m) (s_2^3 - s_1^3) + (3/8 \ k_m^3) (s_2 - s_1)$

(IN 14) $= -(1/4 \ k_m) (s_2^4 - s_1^4) + (3/4 \ k_m^3) (s_2^2 - s_1^2)$

(IN 15) $= \Pi$ and

(IN 16) $= \Pi$